

Phase Retrieval by Linear Algebra

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Abstract

The null vector method, based on a simple linear algebraic concept, is proposed as a solution to the phase retrieval problem. In the case with complex Gaussian random measurement matrices, a non-asymptotic error bound is derived, yielding an asymptotic regime of accurate approximation comparable to that for the spectral vector method.

1 Introduction

We consider the following phase retrieval problem: Let $A = [a_i]$ be a $n \times N$ random matrix with independently and identically distributed entries in $N(0, 1) + iN(0, 1)$, i.e. circularly symmetric complex Gaussian random variables. Let $x_0 \in \mathbb{C}^n$ and $y = A^*x_0$. Suppose we are given A and $b := |y|$ where $|y|$ denote the vector such that $|y|(j) = |y(j)|, \forall j$. The aim of phase retrieval is to find x_0 .

Clearly this is a nonlinear inversion problem. Simple dimension count shows that, for the solution to be unique in general, the number of (nonnegative) data N needs to be at least twice the number n of unknown (complex) components. There are many approaches to phase retrieval, the most efficient and effective, especially when the problem size is large, being fixed point algorithms (see [3, 4, 6, 7] and references therein) and non-convex optimization methods [1, 2]. Phase retrieval has a wide range of applications, see [8] for a recent survey.

The following observation motivates our current approach: Let I be a subset of $\{1, \dots, N\}$ and I_c its complement such that $b(i) \leq b(j)$ for all $i \in I, j \in I_c$. In other words, $\{b(i) : i \in I\}$ are the “weaker” signals and $\{b(j) : j \in I_c\}$ the “stronger” signals. Let $|I|$ be the cardinality of the set I . Since $b(i) = |a_i^*x_0|, i \in I$, are small, $\{a_i\}_{i \in I}$ is a set of sensing vectors nearly orthogonal to x_0 . Denote the sub-column matrices consisting of $\{a_i\}_{i \in I}$ and $\{a_j\}_{j \in I_c}$ by A_I

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and A_{I_c} , respectively. Define the null vector by the singular vector for the least singular value of A_I :

$$x_{\text{null}} := \arg \min \{ \|A_I^* x\|^2 : x \in \mathbb{C}^n, \|x\| = \|x_0\| \}$$

which can be computed by purely linear algebraic methods.

The goal of the paper is to establish a regime where x_{null} is an accurate approximation to x_0 .

2 Approximation theorem

Note that both x_{null} and the phase retrieval solution is at best uniquely defined up to a global phase factor. So we use the following error metric

$$\|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\|^2 = 2\|x_0\|^4 - 2|x_0^* x_{\text{null}}|^2 \quad (1)$$

which has the advantage of being independent of the global phase factor.

The following theorem is our main result.

Theorem 2.1. *Suppose*

$$\sigma := \frac{|I|}{N} < 1, \quad \nu = \frac{n}{|I|} < 1. \quad (2)$$

Then for any $\epsilon \in (0, 1)$, $\delta > 0$ and $t \in (0, \nu^{-1/2} - 1)$ the following error bound

$$\|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\|^2 \leq \left(\left(\frac{2+t}{1-\epsilon} \right) \sigma + \epsilon (-2 \ln(1-\sigma) + \delta) \right) \frac{2\|x_0\|^4}{(1 - (1+t)\sqrt{\nu})^2} \quad (3)$$

holds with probability at least

$$1 - 2 \exp(-N \delta^2 e^{-\delta} |1 - \sigma|^2 / 2) - \exp(-2 \lfloor |I| \epsilon \rfloor^2 / N) - Q \quad (4)$$

where Q has the asymptotic upper bound

$$2 \exp \left\{ -c \min \left[\frac{e^2 t^2}{16} (\ln \sigma^{-1})^2 |I|^2 / N, \frac{et}{4} |I| \ln \sigma^{-1} \right] \right\}, \quad \sigma \ll 1, \quad (5)$$

with an absolute constant c .

Remark 2.2. *To unpack the implications of Theorem 2.1, consider the following asymptotic: With ϵ and t fixed, let*

$$n \gg 1, \quad \sigma = \frac{|I|}{N} \ll 1, \quad \frac{|I|^2}{N} \gg 1, \quad \nu = \frac{n}{|I|} < 1.$$

We have

$$\|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\|^2 \leq c_0 \sigma \|x_0\|^4 \quad (6)$$

with probability at least

$$1 - c_1 e^{-c_2 n} - c_3 \exp \left\{ -c_4 (\ln \sigma^{-1})^2 |I|^2 / N \right\}$$

for moderate constants c_0, c_1, c_2, c_3, c_4 .

The proof of Theorem 2.1 is given in the next section.

The spectral vector method [1, 2, 7] is another linear algebraic method and uses the leading singular vector x_{spec} of $B^* = \text{diag}[b]A^*$ to approximate x_0 where

$$x_{\text{spec}} := \arg \max \left\{ \|B^* x\|^2 : x \in \mathbb{C}^n, \|x\| = \|x_0\| \right\}.$$

The spectral vector method has a comparable performance guarantee to (6) which vanishes as $\sigma \rightarrow 0$, with probability close to 1 exponentially in n .

In practice, however, the null vector method significantly outperforms the spectral vector method in terms of accuracy and noise stability when b is contaminated with noise [4].

The drawback with both approaches is that the error metric vanishes only with infinitely many data, $N \rightarrow \infty$. For a finite data set, the null vector is best to be deployed in conjunction with a fast (locally) convergent fixed point algorithm such as alternating projection [4] or the Douglas-Rachford algorithm [3].

3 Proof of Theorem 2.1

The proof is based on the following two propositions.

Proposition 3.1. *There exists $x_{\perp} \in \mathbb{C}^n$ with $x_{\perp}^* x_0 = 0$ and $\|x_{\perp}\| = \|x_0\| = 1$ such that*

$$\frac{1}{4} \|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\|^2 \leq \frac{\|b_I\|^2}{\|A_I^* x_{\perp}\|^2}. \quad (7)$$

Proof. Since x_{null} is optimally phase-adjusted, we have

$$\beta := x_0^* x_{\text{null}} \geq 0 \quad (8)$$

and

$$x_0 = \beta x_{\text{null}} + \sqrt{1 - \beta^2} z \quad (9)$$

for some unit vector $z^* x_{\text{null}} = 0$. Then

$$x_{\perp} := -(1 - \beta^2)^{1/2} x_{\text{null}} + \beta z \quad (10)$$

is a unit vector satisfying $x_0^* x_\perp = 0$. Since x_{null} is a singular vector and z belongs in another singular subspace, we have

$$\begin{aligned}\|A_I^* x_0\|^2 &= \beta^2 \|A_I^* x_{\text{null}}\|^2 + (1 - \beta^2) \|A_I^* z\|^2, \\ \|A_I^* x_\perp\|^2 &= (1 - \beta^2) \|A_I^* x_{\text{null}}\|^2 + \beta^2 \|A_I^* z\|^2\end{aligned}$$

from which it follows that

$$\begin{aligned}(2 - \beta^2) \|A_I^* x_0\|^2 - (1 - \beta^2) \|A_I^* x_\perp\|^2 \\ = \|A_I^* x_{\text{null}}\|^2 + 2(1 - \beta^2)^2 (\|A_I^* z\|^2 - \|A_I^* x_{\text{null}}\|^2) \geq 0.\end{aligned}\tag{11}$$

By (11), (1) and $\|b_I\| = \|A_I^* x_0\|$, we also have

$$\frac{\|b_I\|^2}{\|A_I^* x_\perp\|^2} \geq \frac{1 - \beta^2}{2 - \beta^2} \geq \frac{1}{2}(1 - \beta^2) = \frac{1}{4} \|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\|^2.\tag{12}$$

□

Proposition 3.2. *Let $A \in \mathbb{C}^{n \times N}$ be an i.i.d. complex standard Gaussian random matrix. Then for any $\epsilon > 0, \delta > 0, t > 0$*

$$\|b_I\|^2 \leq |I| \left(\left(\frac{2+t}{1-\epsilon} \right) \frac{|I|}{N} + \epsilon \left(-2 \ln \left(1 - \frac{|I|}{N} \right) + \delta \right) \right)$$

with probability at least

$$1 - 2 \exp(-N \delta^2 e^{-\delta} |1 - \sigma|^2 / 2) - 2 \exp(-2 \epsilon^2 |1 - \sigma|^2 \sigma^2 N) - Q$$

where Q has the asymptotic upper bound

$$2 \exp \left\{ -c \min \left[\frac{e^2 t^2}{16} \frac{|I|^2}{N} (\ln \sigma^{-1})^2, \frac{et}{4} |I| \ln \sigma^{-1} \right] \right\}, \quad \sigma := \frac{|I|}{N} \ll 1.$$

The proof of Proposition 3.2 is given in the next section.

Now we turn to the proof of Theorem 2.1.

Without loss of the generality we may assume $\|x_0\| = 1$. Otherwise, we replace x_0, x_{null} by $x_0/\|x_0\|$ and $x_{\text{null}}/\|x_0\|$, respectively. Let $Q = [Q_1 \ Q_2 \ \cdots \ Q_n]$ be a unitary transformation where $Q_1 = x_0$ or equivalently $x_0 = Q e_1$ where e_1 is the canonical vector with 1 as the first entry and zero elsewhere. Since unitary transformations do not affect the covariance structure of Gaussian random vectors, the matrix $A^* Q$ is an i.i.d. complex standard Gaussian matrix.

Proposition 3.3. *Let I be any set such that $b(i) \leq b(j)$ for all $i \in I$ and $j \in I_c = \{1, 2, \dots, N\} \setminus I$. For any unitary matrix Q , let $A' \in \mathbb{C}^{|I| \times (n-1)}$ be the sub-column matrix of $A_I^* Q$ with its first column vector deleted. Then A' is an i.i.d. complex standard Gaussian random matrix.*

Proof. First note that $A_I^*Q = (A^*Q)_I$, the row submatrix of A^*Q indexed by I . As noted already, A^*Q is an i.i.d. complex Gaussian matrix.

Since $x_0 = Qe_1$ and $b = |A^*Qe_1|$, I and I_c are entirely determined by the first column of A^*Q which is independent of the other columns of A^*Q . Consequently, the probability law of A' conditioned on the choice of I equals the probability law of A' for a fixed I . Therefore, A' is an i.i.d. complex standard Gaussian matrix. \square

Let $\{\nu_i\}_{i=1}^{n-1}$ be the singular values of A' in the ascending order. For any $z \in \mathbb{C}^{n-1}$,

$$B' := A' \text{diag}(z/|z|)$$

has the same set of singular values as A' . Again, we adopt the convention that $z(j)/|z(j)| = 1$ when $z(j) = 0$. We have

$$\|A'z\| = \|B'|z|\|$$

and hence

$$\|A'z\| = (\|\Re(B')|z|\|^2 + \|\Im(B')|z|\|^2)^{1/2} \geq \sqrt{2}(\|\Re(B')|z|\| \wedge \|\Im(B')|z|\|).$$

By the theory of Wishart matrices [5], the singular values $\{\nu_j^R\}_{j=1}^{n-1}, \{\nu_j^I\}_{j=1}^{n-1}$ (in the ascending order) of $\Re(B'), \Im(B')$ satisfy the probability bounds that for every $t > 0$ and $j = 1, \dots, n-1$

$$\mathbb{P}\left(\sqrt{|I|} - (1+t)\sqrt{n} \leq \nu_j^R \leq \sqrt{|I|} + (1+t)\sqrt{n}\right) \geq 1 - 2e^{-nt^2/2}, \quad (13)$$

$$\mathbb{P}\left(\sqrt{|I|} - (1+t)\sqrt{n} \leq \nu_j^I \leq \sqrt{|I|} + (1+t)\sqrt{n}\right) \geq 1 - 2e^{-nt^2/2}. \quad (14)$$

By Proposition 3.1 and (13)-(14), we have

$$\begin{aligned} \|x_0x_0^* - x_{\text{null}}x_{\text{null}}^*\| &\leq \frac{\sqrt{2}\|b_I\|}{\|\Re(B')|y|\| \wedge \|\Im(B')|y|\|} \\ &\leq \sqrt{2}\|b_I\|(\nu_{n-1}^R \wedge \nu_{n-1}^I)^{-1} \\ &\leq \sqrt{2}\|b_I\|(\sqrt{|I|} - (1+t)\sqrt{n})^{-1}. \end{aligned}$$

By Proposition 3.2, we obtain the desired bound (3). The success probability is at least the expression (13) minus $4e^{-nt^2/2}$ which equals the expression (4).

3.1 Proof of Proposition 3.2

By the Gaussian assumption, $b(i)^2 = |a_i^*x_0|^2$ has a chi-squared distribution with the probability density $e^{-z/2}/2$ on $z \in [0, \infty)$ and the cumulative distribution

$$F(\tau) := \int_0^\tau 2^{-1} \exp(-z/2) dz = 1 - \exp(-\tau/2).$$

Let

$$\tau_* = -2 \ln(1 - |I|/N) \quad (15)$$

for which $F(\tau_*) = |I|/N$.

Define

$$\hat{I} := \{i : b(i)^2 \leq \tau_*\} = \{i : F(b^2(i)) \leq |I|/N\},$$

and

$$\|\hat{b}\|^2 := \sum_{i \in \hat{I}} b(i)^2.$$

Let

$$\{\tau_1 \leq \tau_2 \leq \dots \leq \tau_N\}$$

be the sorted sequence of $\{b(1)^2, \dots, b(N)^2\}$ in magnitude.

Proposition 3.4. (i) *For any $\delta > 0$, we have*

$$\tau_{|I|} \leq \tau_* + \delta \quad (16)$$

with probability at least

$$1 - \exp\left(-\frac{N}{2} \delta^2 e^{-\delta} |1 - |I|/N|^2\right) \quad (17)$$

(ii) *For each $\epsilon > 0$, we have*

$$|\hat{I}| \geq |I|(1 - \epsilon) \quad (18)$$

or equivalently,

$$\tau_{\lfloor |I|(1-\epsilon) \rfloor} \leq \tau_* \quad (19)$$

with probability at least

$$1 - 2 \exp(-4\epsilon^2 |1 - |I|/N|^2 |I|^2/N) \quad (20)$$

Proof. (i) Since $F'(\tau) = \exp(-\tau/2)/2$,

$$|F(\tau + \epsilon) - F(\tau)| \geq \epsilon/2 \exp(-(\tau + \epsilon)/2). \quad (21)$$

For $\delta > 0$, let

$$\zeta := F(\tau_* + \delta) - F(\tau_*)$$

which by (21) satisfies

$$\zeta \geq \frac{\delta}{2} \exp(-\frac{1}{2}(\tau_* + \delta)). \quad (22)$$

Let $\{w_i : i = 1, \dots, N\}$ be the i.i.d. indicator random variables

$$w_i = \chi_{\{b(i)^2 > \tau_* + \delta\}}$$

whose expectation is given by

$$\mathbb{E}[w_i] = 1 - F(\tau_* + \delta).$$

The Hoeffding inequality yields

$$\begin{aligned} \mathbb{P}(\tau_{|I|} > \tau_* + \delta) &= \mathbb{P}\left(\sum_{i=1}^N w_i > N - |I|\right) \\ &= \mathbb{P}\left(N^{-1} \sum_{i=1}^N w_i - \mathbb{E}[w_i] > 1 - |I|/N - \mathbb{E}[w_i]\right) \\ &= \mathbb{P}\left(N^{-1} \sum_{i=1}^N w_i - \mathbb{E}[w_i] > \zeta\right) \\ &\leq \exp(-2N\zeta^2). \end{aligned} \tag{23}$$

Hence, for any fixed $\delta > 0$,

$$\tau_{|I|} \leq \tau_* + \delta \tag{24}$$

holds with probability at least

$$\begin{aligned} 1 - \exp(-2N\zeta^2) &\geq 1 - \exp\left(-\frac{N\delta^2}{2}e^{-\tau_* - \delta}\right) \\ &= 1 - \exp\left(-\frac{N\delta^2}{2}e^{-\delta}|1 - |I|/N|^2\right) \end{aligned}$$

by (22).

(ii) Consider the following replacement

$$\begin{aligned} (a) \quad &|I| \longrightarrow \lceil |I|(1 - \epsilon) \rceil \\ (b) \quad &\tau_* \longrightarrow F^{-1}(\lceil |I|(1 - \epsilon) \rceil / N) \\ (c) \quad &\delta \longrightarrow F^{-1}(|I|/N) - F^{-1}(\lceil |I|(1 - \epsilon) \rceil / N) \\ (d) \quad &\zeta \longrightarrow F^{-1}(\tau_* + \delta) - F^{-1}(\tau_*) = |I|/N - \lceil |I|(1 - \epsilon) \rceil / N = \frac{\lfloor |I|\epsilon \rfloor}{N} \end{aligned}$$

in the preceding argument. Then (23) becomes

$$\mathbb{P}(\tau_{\lceil |I|(1 - \epsilon) \rceil} > F^{-1}(|I|/N)) \leq \exp(-2N\zeta^2) = \exp\left(-\frac{2\lfloor |I|\epsilon \rfloor^2}{N}\right).$$

That is,

$$\tau_{\lceil |I|(1 - \epsilon) \rceil} \leq \tau_*$$

holds with probability at least

$$1 - \exp(-2\lfloor |I|\epsilon \rfloor^2/N).$$

□

Proposition 3.5. *For each $\epsilon > 0$ and $\delta > 0$,*

$$\frac{\|b_I\|^2}{|I|} \leq \frac{\|\hat{b}\|^2}{|\hat{I}|} + \epsilon(\tau_* + \delta) \quad (25)$$

with probability at least

$$1 - 2 \exp\left(-\frac{1}{2}\delta^2 e^{-\delta} |1 - |I|/N|^2 N\right) - 2 \exp\left(-2\epsilon^2 |1 - |I|/N|^2 \frac{|I|^2}{N}\right). \quad (26)$$

Proof. Since $\{\tau_j\}$ is an increasing sequence, the function $T(m) = m^{-1} \sum_{i=1}^m \tau_i$ is also increasing. Consider the two alternatives either $|I| \geq |\hat{I}|$ or $|\hat{I}| \geq |I|$. For the latter,

$$\|b_I\|^2/|I| \leq \|\hat{b}\|^2/|\hat{I}|$$

due to the monotonicity of T .

For the former case $|I| \geq |\hat{I}|$, we have

$$\begin{aligned} T(|I|) &= |I|^{-1} \sum_{i=1}^{|\hat{I}|} \tau_i + |I|^{-1} \sum_{i=|\hat{I}|+1}^{|I|} \tau_i \\ &\leq T(|\hat{I}|) + |I|^{-1}(|I| - |\hat{I}|)\tau_{|I|}. \end{aligned}$$

By Proposition 3.4 (ii) $|\hat{I}| \geq (1 - \epsilon)|I|$ and hence

$$T(|I|) \leq T(|\hat{I}|) + |I|^{-1}(|I| - |I|(1 - \epsilon))\tau_{|I|} = T(|\hat{I}|) + \epsilon\tau_{|I|}$$

with probability at least given by (20).

By Proposition 3.4 (i), $\tau_{|I|} \leq \tau_* + \delta$ with probability at least given by (17). □

Continuing the proof of Proposition 3.2, let us consider the i.i.d. centered, bounded random variables

$$Z_i := \frac{N^2}{|I|^2} [b(i)^2 \chi_{\tau_*} - \mathbb{E}[b(i)^2 \chi_{\tau_*}]] \quad (27)$$

where χ_{τ_*} is the characteristic function of the set $\{b(i)^2 \leq \tau_*\}$. Note that

$$\mathbb{E}(b(j)^2 \chi_{\tau_*}) = \int_0^{\tau_*} 2^{-1} z \exp(-z/2) dz = 2 - (\tau_* + 2) \exp(-\tau_*/2) \leq 2|I|^2/N^2 \quad (28)$$

and hence

$$-2 \leq Z_i \leq \sup \left\{ \frac{N^2}{|I|^2} b(i)^2 \chi_{\tau_*} \right\} = \frac{N^2}{|I|^2} \tau_*. \quad (29)$$

Next recall the Bernstein-inequality.

Proposition 3.6. [9] *Let Z_1, \dots, Z_N be i.i.d. centered sub-exponential random variables. Then for every $t \geq 0$, we have*

$$\mathbb{P} \left\{ N^{-1} \left| \sum_{i=1}^N Z_i \right| \geq t \right\} \leq 2 \exp \left\{ -c \min(Nt^2/K^2, Nt/K) \right\}, \quad (30)$$

where c is an absolute constant and

$$K = \sup_{p \geq 1} p^{-1} (\mathbb{E} |Z_j|^p)^{1/p}.$$

Remark 3.7. *For K we have the following estimates*

$$\begin{aligned} K &\leq \frac{2N^2}{|I|^2} \sup_{p \geq 1} p^{-1} (\mathbb{E} |b(i)^2 \chi_{\tau_*}|^p)^{1/p} \\ &\leq \frac{2N^2}{|I|^2} \tau_* \sup_{p \geq 1} p^{-1} (\mathbb{E} \chi_{\tau_*})^{1/p} \\ &\leq \frac{2N^2}{|I|^2} \tau_* \sup_{p \geq 1} p^{-1} (1 - e^{-\tau_*/2})^{1/p}. \end{aligned} \quad (31)$$

The maximum of the right hand side of (31) occurs at

$$p_* = -\ln(1 - e^{-\tau_*/2})$$

and hence

$$K \leq \frac{2N^2}{|I|^2} \frac{\tau_*}{p_*} (1 - e^{-\tau_*/2})^{1/p_*}.$$

We are interested in the regime

$$\tau_* \asymp 2|I|/N \ll 1$$

which implies

$$p_* \asymp -\ln \frac{\tau_*}{2} \asymp \ln \frac{N}{|I|}$$

and consequently

$$K \leq \frac{4N}{e|I|} \left(\ln \frac{N}{|I|} \right)^{-1}, \quad \sigma = |I|/N \ll 1. \quad (32)$$

On the other hand, upon substituting the asymptotic bound (32) in the probability bound

$$Q = 2 \exp \left\{ -c \min(Nt^2/K^2, Nt/K) \right\}$$

of (30), we have

$$K \leq 2 \exp \left\{ -c \min \left[\frac{e^2 t^2}{16} (\ln \sigma^{-1})^2 |I|^2/N, \frac{et}{4} |I| \ln \sigma^{-1} \right] \right\}, \quad \sigma \ll 1.$$

The Bernstein inequality ensures that with high probability

$$\left| \frac{\|\hat{b}\|^2}{N} - \mathbb{E}(b^2(i)\chi_{\tau_*}) \right| \leq t \frac{|I|^2}{N^2}.$$

By (18) and (28), we also have

$$\begin{aligned} \frac{\|\hat{b}\|^2}{|\hat{I}|} &\leq \mathbb{E}(b(i)^2 \chi_{\tau_*}) \frac{N}{|\hat{I}|} + t \frac{|I|^2}{|\hat{I}|N} \\ &\leq \left(\mathbb{E}(b(i)^2 \chi_{\tau_*}) \frac{N^2}{|I|^2} + t \right) \frac{|I|}{N} \\ &\leq \frac{2+t}{1-\epsilon} \cdot \frac{|I|}{N} \end{aligned} \tag{33}$$

By Prop. 3.5, we now have

$$\|b_I\|^2 \leq |I| \left(\frac{\|\hat{b}\|^2}{|\hat{I}|} + \epsilon(\tau_* + \delta) \right)$$

with probability at least given by (4), which together with (33) and (15) complete the proof of Proposition 3.2.

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